Who says that an identity cannot be proved simply by substitution

Yue Kwok Choy

Can we prove an identity, say $x^3 - 1 = (x-1)(x^2+x+1)$ just be substituting some numerical values of x? To many surprise, the answer is **YES**.

Let $p(x) = x^3 - 1 - (x-1)(x^2+x+1)$

$p(0) = 0^3 - 1 - (0-1)(0^2+0+1) = 0$

$p(1) = 1^3 - 1 - (1-1)(1^2+1+1) = 0$

$p(2) = 2^3 - 1 - (2-1)(2^2+2+1) = 0$

$p(3) = 3^3 - 1 - (3-1)(3^2+3+1) = 0$

Now, the degree of $p(x)$ is THREE but there are FOUR points which satisfy the polynomial.

Therefore, $p(x) \equiv 0$ and $x^3 - 1 = (x-1)(x^2+x+1)$!

More examples with increasing complexity:

(1) Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$, for natural numbers n.

Let $p(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2 - \frac{1}{6}n(n+1)(2n+1)$

$p(1) = 1^2 - \frac{1}{6}(1)(2)(3) = 0$

$p(2) = 1^2 + 2^2 - \frac{1}{6}(2)(3)(5) = 0$

$p(3) = 1^2 + 2^2 + 3^2 - \frac{1}{6}(3)(4)(7) = 0$

$p(4) = 1^2 + 2^2 + 3^2 + 4^2 - \frac{1}{6}(4)(5)(9) = 0$

Now, the degree of $p(x)$ is THREE but there are FOUR points which satisfy the polynomial.

Therefore, $p(x) \equiv 0$ and $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

Exercise Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{1}{2}n(n+1)\right]^2$, for natural numbers n.

(2) Prove the identity:

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2\frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2\frac{(x-a)(x-b)}{(c-a)(c-b)} = x^2$$

Let $p(x) = a^2\frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2\frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2\frac{(x-a)(x-b)}{(c-a)(c-b)} - x^2$

Then $p(a) = p(b) = p(c) = 0$ (Please check yourselves.)

Now, the degree of $p(x)$ is TWO but there are THREE points which satisfy the polynomial.

Therefore, $p(x) \equiv 0$ and $a^2\frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2\frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2\frac{(x-a)(x-b)}{(c-a)(c-b)} = x^2$.